

# Exchanging Goods Using Valuable Money

J. V. Howard\*

27 October, 2015

## Abstract

A group of people wishes to use money to exchange goods efficiently over several time periods. However, there are disadvantages to using any of the goods as money, and in addition fiat money issued in the form of notes or coins will be valueless in the final time period, and hence in all earlier periods. Also, Walrasian market prices are determined only up to an arbitrary rescaling. Nevertheless we show that it is possible to devise a system which uses money to exchange goods and in which money has a determinate positive value. The mechanism controls the flow rather than the stock of money: it introduces some trading frictions, some redistribution of wealth, and some distortion of prices, but these effects can all be made small.

**JEL Codes:** D47, D82.

## 1 Introduction

A fundamental problem in economics is to describe how a group of people could exchange goods or services efficiently to their mutual benefit. Goods can be sold by auction, but that requires money. General equilibrium theory (Walras, Arrow-Debreu) shows how markets can work to exchange homogeneous and divisible goods, but the basic theory has no place for money. However, we observe in practice that money is extremely useful in decentralising economic transactions in both space and time. Hahn (1965) asked whether it is possible to construct GE models where money has to have a positive value in any equilibrium solution (Hahn's problem).

In essence, the problem arises because traders are prepared to exchange goods or services for money only because they believe that in the future they will be able to exchange the money for other goods or services. This implies that fiat money can work as a medium of exchange only when traders can use money to buy something of value from the government (so that money is cleared from the economy). Otherwise, if there were only a (known) finite number of time periods, money would be valueless on the last day, and hence on all days. Even when time is endless, there will still be equilibria in which the value of money is zero.

---

\*Department of Mathematics, London School of Economics, UK. [j.v.howard@lse.ac.uk](mailto:j.v.howard@lse.ac.uk)

Economists have devised models in which money has positive value (see Section 6), but in this paper we will take an alternative approach. We will think of the problem as one of mechanism design or economic engineering, and so we will try to design an economic system in which money is used in all trades, and where the value of money can be controlled by society from one time period to another. We believe that even a single very simplified model showing how such a system could work (proof of concept) is of theoretical interest. We aim to define such a system, and then discuss briefly how it relates to other work.

One possible story would be about a group of families who help each other with baby-sitting, taking the children to school, car sharing for commuting, and so on. Some people begin to suspect that others are wilfully exploiting the community by taking out more than they contribute. So the idea is proposed of introducing some sort of local currency within the group to keep track of debts and credits. But this scenario is still not sufficiently well defined for our purposes. We are looking for an even simpler (if less realistic) setting.

So let us imagine that after a shipwreck a group of people is stranded on a desert island. Fortunately everyone is unharmed, and all escape from the ship with a box of perishable food. They establish contact with the rest of the world, who inform them that they will be rescued by boat in one or two week's time. If they have to stay a second week, fresh food boxes will be dropped by parachute. Meanwhile they are invited to enjoy their enforced holiday, and share their food.

However, they have all acquired a strong feeling of property rights over their own food boxes. Some have a good mixture of items, but others have larger quantities of only one or two fruits or vegetables. As people's tastes and appetites differ, there are obvious benefits to trade, but they have no common currency. They decide it would be an interesting exercise to devise a system for exchanging the foodstuffs, a system which will start and end on the island, and leave no outstanding debts when they leave.

They reject almost immediately the idea of looking for a rare object (perhaps an unusual shell) to be used as money: there would be too much danger that someone might find a place where the shells were plentiful, thereby becoming an instant millionaire and also debasing the currency. Nominating one foodstuff to be used as the medium of exchange is considered more seriously, but there seems no suitable choice. Giving everyone a starting stock of centrally produced banknotes would not work: nobody would want to be left holding the notes at the end of the week (when they would be worthless), so prices would be bid up indefinitely in the markets. However, they know there is another way of creating money: debt.

They understand that it should be possible for the traders themselves to create money by writing promissory notes. But IOU notes for what? Many countries once traded using IOU notes for gold. But, since leaving the gold standard, countries like the US and the UK effectively issue notes for an imaginary commodity (dollars or pounds). They decide to follow this route.

So the newly created finance committee announces that the island's currency unit will be the groat, and that they expect a groat to be about the price of an apple or a

pear. They plan to invite those who wish to trade to take their goods to market, where some sort of Walrasian auction process will take place, leading to suggested prices for all goods. Traders will then be asked to create money by writing and signing IOU notes for one groat. Well known traders in good standing will have their notes accepted by everyone, and so paper money will be created. (In Scotland three Scottish banks as well as the Bank of England can issue banknotes.) In fact we will assume that all traders are in good standing, and so everyone can create money. Trading will then take place using these notes as money, until everyone has the goods they want, plus a bundle of other people's IOU notes.

At times during the trading period a person may have accumulated more debts than credits, but he must ensure that by the close of the period he can match his debts (his IOU notes) with credits (other people's notes). In fact, everybody must arrange to finish trading with their transactions exactly balanced. Then, at the close of trade, outstanding debts must be reconciled.

So a trader with an IOU note will visit the issuer of the note and demand payment. In general he will just be given in exchange another note issued by someone else, but after a number of rounds he should receive one of his own notes, which he then destroys. Assuming that everyone trades honestly, all debts will eventually clear, and all the money created to facilitate trade will be destroyed.

However, this system is vulnerable to cheating. Supposing a trader pays off his debt by simply writing a fresh IOU note (in fact by behaving like many of the world's central banks). The finance committee could make a rule saying that debts must be paid using another person's notes, but then there could be scams involving two traders acting in collusion. The committee decides that the first day will be allowed for trading, and that after that day nobody can issue more notes. The process of resolving debt is then grounded.

We will call this simple system MaDFIC (money as debt for an imaginary commodity). It seems workable (with suitable penalties for cheating), but it determines only relative prices. (If the prices of all goods were doubled, there would be an equally good equilibrium, probably requiring the creation of about twice as much money.) We could try to fix prices by demanding that the IOU notes must be written for one particular good (e.g. apples), but when we come to look at two period problems, there may not be any apples in the second period, while we want the general value of money to remain much the same.

In the extreme version of this scheme, everyone starts by creating as much money as he needs for all his purchases. He then trades, buying goods with his money and receiving other people's money for his sales. Each note is used exactly once on day one, and all notes are destroyed on day two.

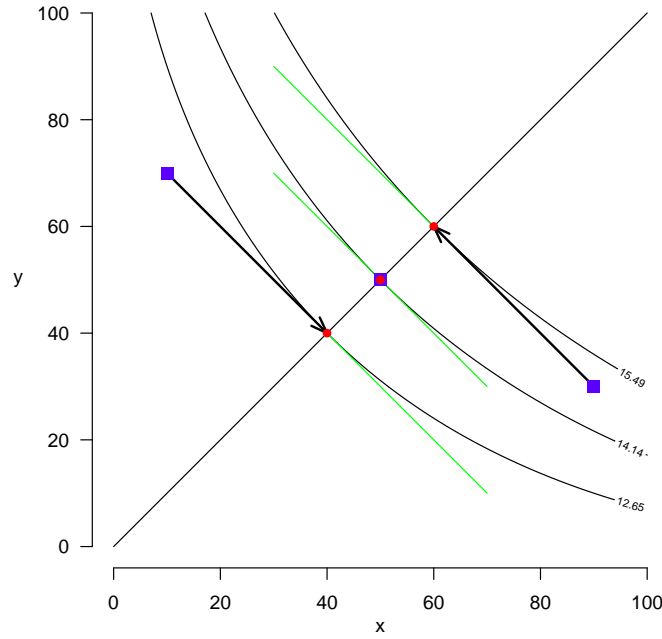
We will now give a trivial numerical example which we will modify later when we see how we might give money a determinate value.

**Example 1.** *We suppose that there are equal large numbers of three types of trader A (the rich), C (the poor), and B (the middle class), and just two goods (apples and pears). Type A starts with 90 apples and 30 pears. Type C starts with 10 apples and 70 pears.*

Type B starts with 50 of each. They all have the same utility function of their holdings of apples,  $x$ , and pears,  $y$ .

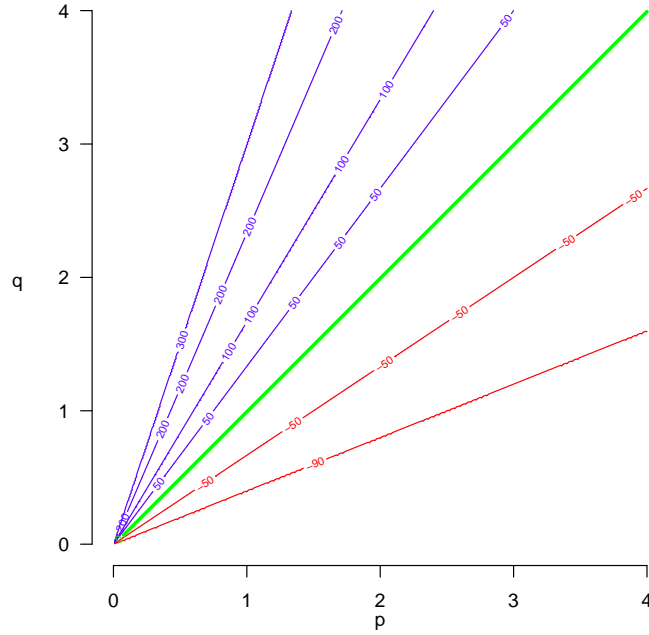
$$u(x, y) = \sqrt{x} + \sqrt{y}.$$

So the rich start with a utility of 15.0, the poor with 11.5, and the middle class with 14.1. In a market equilibrium, apples and pears will have the same price, and A (types) will buy 30 pears from C (types) and sell 30 apples to them, until everyone has equalised their holdings of the two fruits. A's utility will then be 15.5, C's will be 12.6, and B will remain at 14.1. Figure 1 shows these movements in quantity space. A trader starts at a blue square and moves along his green budget constraint to maximise his utility at the red circle.



**Figure 1:** Movements to market equilibrium

Let the prices of apples and pears be  $p$  and  $q$  respectively. Figure 2 shows contours of the excess demand for apples in price space. (Blue contours, with slope greater than 1, have positive excess demand; red contours, with slope less than 1, negative.) On the green diagonal  $p = q$  the supply of apples equals the demand. If we plotted the excess demand for pears, the contour of zero excess demand would also be  $p = q$ . The two equations in  $p$  and  $q$  determine only the ratio  $p : q$ .



**Figure 2:** Excess demand contours

So we have a simple system based on traders creating their own money, IOU notes for groats. There are no real groats. Apart from the initial suggestion for the approximate value of a groat, the finance committee has no further influence on the general price level.

## 2 Introducing tokens

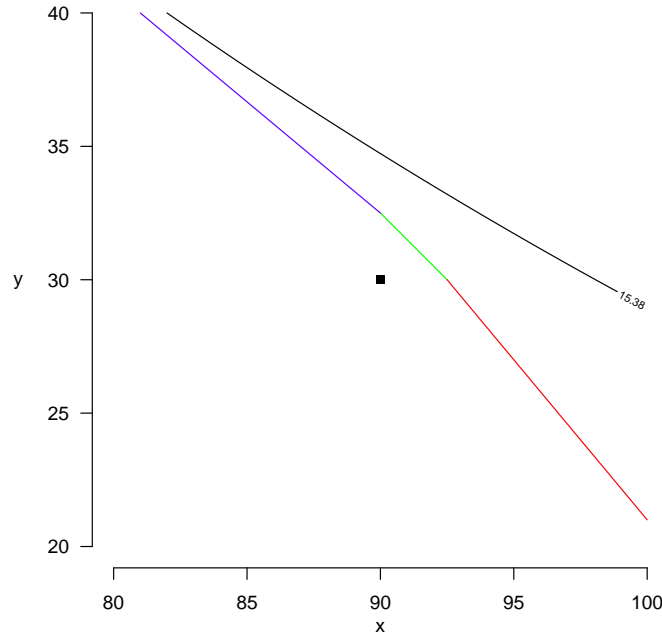
In the proposed financial system the quantity of money is not fixed: money is created by the traders as required, and then destroyed again after trade is finished. So the idea of controlling the absolute price level by restricting the stock of money would not seem feasible. But what about the *flow* of money? Could we control that, and hence give money an absolute value?

Suppose the finance committee sets up a central bank which then gives each trader a fixed number ( $n$ ) of tokens. Whenever he spends a groat on food, he must return a token to the bank. Traders can exchange tokens for groats (or vice versa) without returning any tokens to the bank. All trades must be made using IOU notes for groats. There will clearly be equilibria with low prices where there are more tokens available in total than are required for all the desired trades: the market price of tokens will then be zero. There will also be equilibria in which tokens have a positive value. Then people who want to make few trades will sell tokens to others who wish to make many trades. The total money value of trades is fixed, but there will still be different possible price levels.

**Example 2.** *Continuing with the situation in Example 1, suppose now that before trading all traders are given  $n = 30$  tokens, and instructed to return one token to the bank for each groat spent on goods.*

There are still many equilibria. For example, if apples and pears are priced at  $1/2$  groat, we can have the same equilibrium as before with  $A$  and  $C$  each returning 15 of their 30 tokens to the bank. The same is true for all prices (assumed equal) up to 1 groat. At still higher prices  $A$  and  $C$  will have to purchase tokens from  $B$ , but they will be able to do this at zero cost as long as there is an overall surplus of tokens (when the market price of tokens will be zero). At a common price of 1.5 groats,  $A$  and  $C$  can each buy 30 fruits from the other, buy 15 tokens from  $B$  at zero cost, and give the bank 45 tokens. If the price vector for apples, pears, and tokens is  $(p, q, r)$ , we have equilibrium vectors  $\lambda(1, 1, 0)$  for  $0 < \lambda \leq 1.5$ .

Now suppose that the price of the fruits is (say) 2 groats and the price of a token is  $1/5$  groat. Consider a type  $A$  trader who starts with 90 apples and 30 pears. Her tokens are worth  $rn = 6$  groats, so she could consider buying more fruit. Because she has to return tokens to the bank (effectively a 20% purchase tax), she sees a price of 2.4 rather than 2, so she can buy  $5/2$  fruits, paying 5 groats (which she gets from selling 25 tokens) and keeping 5 tokens (value 1 groat) to pay the bank. The green line in Figure 3 shows the possibilities using this strategy.



**Figure 3:** Budget constraints using tokens

She next considers selling  $-\delta x$  apples and buying as many pears as she can with the proceeds. After the sale she will have capital worth  $6 - 2\delta x$  groats, so she can buy  $\delta y = (6 - 2\delta x)/2.4$  pears. Her possible trades must satisfy  $2\delta x + 2.4\delta y = 6$ . So if she finishes with holdings of  $(x, y)$  we must have that

$$\begin{aligned} 2(x - 90) + 2.4(y - 30) &= 6 \\ 2x + 2.4y &= 258. \end{aligned}$$

The blue line in Figure 3 shows these possibilities.

Finally, she could sell  $-\delta y$  pears and buy as many apples as she could with the proceeds. After the sale she would have capital worth  $6 - 2\delta y$  groats, so she could buy  $\delta x = (6 - 2\delta y)/2.4$  pears. Her possible trades must satisfy  $2.4\delta x + 2\delta y = 6$ . So if she planned to finish with holdings of  $(x, y)$  we must have that

$$\begin{aligned} 2.4(x - 90) + 2(y - 30) &= 6 \\ 2.4x + 2y &= 282 . \end{aligned}$$

These possible outcomes lie on the red line in Figure 3.

In the general case, if a trader purchases  $\delta x_i$  of good  $i$  ( $i = 1, \dots, k$ ) at unit price  $p_i$ , before tokens are introduced her budget constraint would be

$$\sum_i p_i \delta x_i \leq 0 .$$

After tokens are introduced with a price of  $r$ , she will have  $2^k - 1$  constraints, one for each non-empty set  $S \subseteq \{1, \dots, k\}$ , of the form

$$\sum_{i \in S} p_i \delta x_i (1 + \phi_i r) \leq r n ;$$

where  $\phi_i$  is an indicator variable for membership of  $S$  (the set of goods the trader wishes to buy). Since the budget set (assuming free disposal) is convex and her preferences are strictly convex, there will be a unique optimum point for the trader at the given prices.

The black utility contour shown in the figure is the highest Player  $A$  can attain. In fact, simple calculations show that she will sell 19.64 apples and buy 18.86 pears. She receives 39.27 groats for the apples, and she started with tokens worth 6 groats (suppose she exchanges them for groats). The pears cost her 37.73 groats (paid to the seller), but she must give 37.73 tokens (worth 7.55 groats) to the bank. She buys an extra 7.73 tokens for 1.55 groats. Her budget is balanced. Her demands for apples, pears, and tokens are shown in the second column of Table 1. The demands by value are shown in the corresponding column of Table 2.

**Table 1:** Demand (units) for goods by trader

Good	Trader			Total
	$A$	$B$	$C$	
Apples	-19.64	1.25	22.20	3.81
Pears	18.86	1.25	-23.64	-3.52
Tokens	7.73	-25.00	14.39	-2.88

**Table 2:** Demand (groats) for goods by trader

Good	Trader			Total
	A	B	C	
Apples	-39.27	2.50	44.39	7.62
Pears	37.73	2.50	-47.27	-7.05
Tokens	1.55	-5.00	2.88	-0.58
Total	0	0	0	0

In a similar way, *C* will sell 23.64 pears and buy 22.20 apples and 14.39 tokens.

*B* will clearly use his 30 tokens (worth 6 groats) to buy  $3/2.4 = 1.25$  apples and 1.25 pears. He pays 5 groats for the fruit and gives 5 tokens to the bank.

We now see that at the assumed prices, the total demand for apples is  $22.20 + 1.25 = 23.45$ . The total supply is 19.64. The price of apples will tend to rise. The supply of pears exceeds the demand: the price of pears will tend to fall. The total demand for tokens is 22.12 and the total supply is 25. The price of tokens will tend to fall.

We look for an equilibrium keeping the price of tokens,  $r$ , at  $1/5$ . We find that we must have  $p = 2.075$  and  $q = 2.022$ .

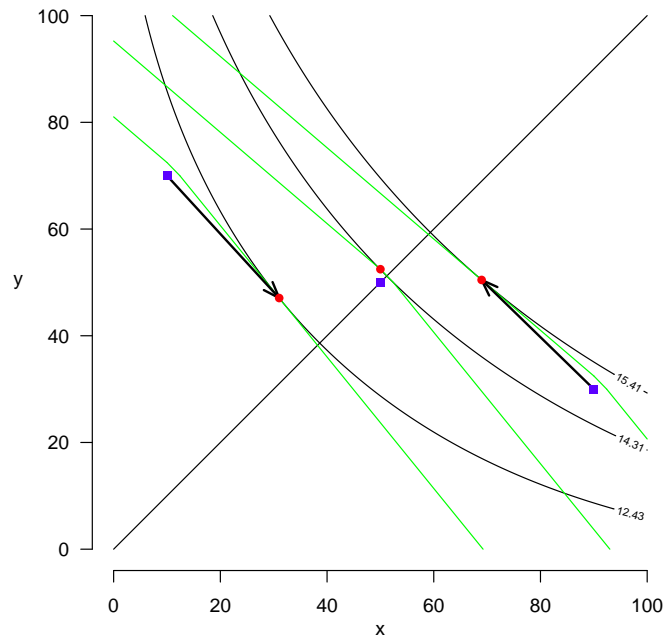
We then find the revised demands for goods shown in Table 3.

**Table 3:** Equilibrium demands (groats) for goods by trader

Good	Trader			Total
	A	B	C	
Apples	-43.64	0.00	43.64	0
Pears	41.36	5.00	-46.36	0
Tokens	2.27	-5.00	2.73	0
Total	0	0	0	0

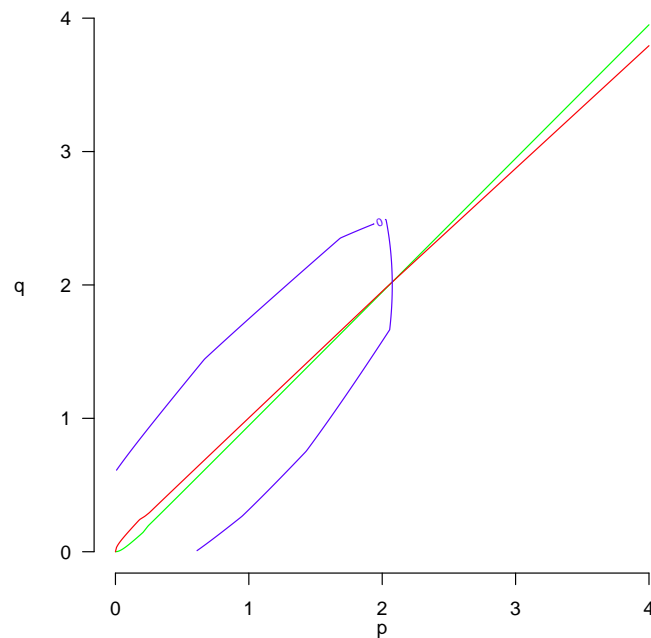
In this solution, trader *B* maximises his utility at a corner point of his budget set (he sells tokens and buys pears). We can recalculate Figure 1.





**Figure 4:** Revised movements to market equilibrium

If we look at the contour lines where supply equals demand for the three markets (apples, pears, and tokens), we find Figure 5. The green and red lines show where the apple and pear markets (respectively) are in equilibrium, and the irregular blue oval is for the token market.



**Figure 5:** Equilibria for three markets

This second system is based on (real) tokens and IOU notes for groats. (As before, there are no real groats.) The central bank is now involved because it issues and receives the tokens. It also has to decide how many tokens,  $n$ , each trader should be

given as her initial endowment. However, prices are still not fixed. It seems that when the price of tokens is zero we will have an infinity of possible price vectors, and we may also be able to find (by choosing different values for  $r$ ) an infinite set of price vectors with  $r > 0$ . How can we fix the value of  $r$ ?

### 3 Fixing the price of tokens

One way of avoiding the problem of multiple equilibria with low price levels (and tokens having zero value) would be for the bank to print money and give each trader a fixed amount of money as well as his allowance of tokens. Traders would still be allowed to create money, but bank money would always be legal tender for making purchases or clearing debt. So this system would have real money, IOU notes, and tokens. Now nobody would want to be left holding bank money, so prices would rise without limit. The bank needs to have something to offer in exchange for its money (other than more money). The obvious answer is tokens.

So suppose the bank offers tokens for money at a fixed exchange rate. Then as soon as the excess money supply has driven prices up to the point where tokens become as valuable as the official bank price, traders will use their money to buy tokens until all bank money disappears from the economy, and a definite price level will be determined.

But this has now become unnecessarily complicated. If tokens and money are to be exchangeable at a fixed rate, why have both? The central bank could simply give each trader a fixed number of tokens. He can then trade using tokens or promissory notes for tokens. Whenever he buys goods he must pay the bank a given percentage, for example 10%, of the the amount he gives the seller (i.e. there is a purchase tax). At the end of the first day, traders will have debts, credits, and tokens: the bank will have only credits for tokens. If all accounts balance, on the second day all the tokens will be returned to the bank and all the IOU notes will be returned to their issuer to be destroyed.

In this system, effectively the finance committee and the central bank impose a purchase tax and then divide the proceeds equally amongst the citizens – although in fact the bank pays out the proceeds before collecting the tax. We have already calculated the equilibrium solution for our example if the tax rate is 20%. In general, the tax will have three effects:

- (a) it is somewhat redistributive as everyone receives the same income from the bank;
- (b) people who wish to trade less than average are at an advantage because they have surplus tokens to sell;
- (c) all buyers face higher prices, so trade will be restricted (as with any purchase tax).

In our example without tokens, apples and pears are equally priced, so we can measure wealth by total holdings of fruit. Table 4 shows how wealth is changed by the system. *A* and *C* lose because they wish to trade more than the average, and *B* gains because he does not want to trade. In the example, this effect is dominant.

**Table 4:** Total holdings of goods

Situation	Trader		
	<i>A</i>	<i>B</i>	<i>C</i>
At start	120·0	100·0	80·0
At end when $r = 0$	120·0	100·0	80·0
At end when $r = 20\%$	119·4	102·5	78·1

Of course the theory assumes that traders care about utility rather than wealth. Table 5 shows how trade increases utility.

**Table 5:** Utility changes

Situation	Trader		
	<i>A</i>	<i>B</i>	<i>C</i>
At start	14·96	14·14	11·53
At end when $r = 0$	15·49	14·14	12·65
At end when $r = 20\%$	15·41	14·31	12·43

This third and final system is based on real tokens and IOU notes for tokens. We will call it MaDFoRT (money as debt for real tokens). Instead of a gold standard, we have a token standard. The groat is now redefined as  $1/r$  tokens. (The advantage of retaining the groat is that when  $r \rightarrow 0$  from above, the prices of commodities in tokens tend to infinity, but prices in groats tend to fixed values, which are market clearing prices in the system without tokens.) Like gold, we assume tokens are kept in a secure vault, and all trading on the first day is done using promissory notes. The bank also receives its purchase tax in the form of notes on day one. On day two, all credits and debts are reconciled, and the bank gets back all its tokens from the vaults. The finance committee now has two parameters under its control:  $r$ , the purchase tax rate, and  $n$  the initial endowment of tokens for each trader. Equilibrium prices (if they exist) will be directly proportional to  $n$ . (In our example  $r = 1/5$  and  $n = 30$ .) It seems this system can control prices, but will there always be an equilibrium solution?

### 3.1 Existence of equilibrium

Suppose there is a market equilibrium without using tokens (i.e. when  $r = 0$ ). Will there always be an equilibrium when  $r$  is given a definite value, say 10%? The answer is clearly no. Modify our example so that the traders all start with equal amounts of

the two goods (so  $A$  might start with 60 apples and 60 pears,  $B$  with 50 of each, and  $C$  with 40 of each). There is an equilibrium with equal prices,  $p = q$ , in which no one wishes to trade. But as soon as tokens are introduced with  $r > 0$ , prices will be bid up indefinitely. There is no equilibrium short of  $p = q = \infty$ . If we keep  $r > 0$  fixed, we could find various starting positions for the 3 traders near the main diagonal  $y = x$  where there would be no equilibrium.

However, if we start with a situation where there is an equilibrium involving some trading when  $r = 0$ , then under reasonable continuity assumptions we would expect to have a similar equilibrium for sufficiently small  $r > 0$ . Also, for sufficiently small  $r$ , we should be able to make price distortions and redistribution effects as small as we like. In this sense, the proposed mechanism does formally solve Hahn's problem.

### 3.2 The need for debt

Now we have a system with real money (in the form of tokens), do we still need extra money (in the form of debt as promissory notes)? We could imagine everyone starts with an account with a balance of 30 tokens, and a debit card allowing payments whilst the account is in credit. Then, in our example with  $r = 1/5$ , trader  $B$  can simply use his debit card firstly to buy pears from  $C$  using five-sixths of his 30 tokens, and then to pay the remainder of the tokens as tax to the bank. Then he has completed all his trades and his account is empty.  $C$  now has 55 tokens.

But life is not so simple for  $A$  and  $C$ . Suppose  $A$  spends five-sixths of his 30 on buying pears from  $C$ , and then pays the rest as tax.  $A$  now has no tokens and  $C$  has 80. Next  $C$  can use all his tokens to buy apples from  $A$  (paying the appropriate amount of tax): then  $A$  can buy more pears from  $C$ , and so on. After an infinite number of transactions,  $A$  will have spent  $2275/11$  tokens buying pears from  $C$  (and paid  $455/11$  tokens in tax).  $C$  will have spent  $2400/11$  tokens buying apples from  $A$  (and paid  $480/11$  tokens in tax). Eventually the bank gets 85 tokens back from  $A$  and  $C$  together, and already has 5 from  $B$ , so all the original 90 tokens are returned, and no account is ever in debt.

However, if the traders are issued with credit cards instead of debit cards, after  $B$  has bought pears from  $C$ ,  $A$  can use credit to spend  $2275/11$  tokens on pears from  $C$  and to pay  $480/11$  tokens in tax. She then has debt of  $2400/11$  (approximately 218) tokens. Finally  $C$  (who now has  $2880/11$  tokens) spends it all on apples from  $A$  and tax. All accounts are now zero.

We see that in order to have enough liquidity in the system, it is still essential to allow traders to create money in the form of debt.

### 3.3 Taxation and public spending

So far we have envisaged a central bank but no government spending. (The redistributive effect of the token system should be thought of as a by-product of the need to maintain a stable currency, not as a social policy.) However, suppose a government

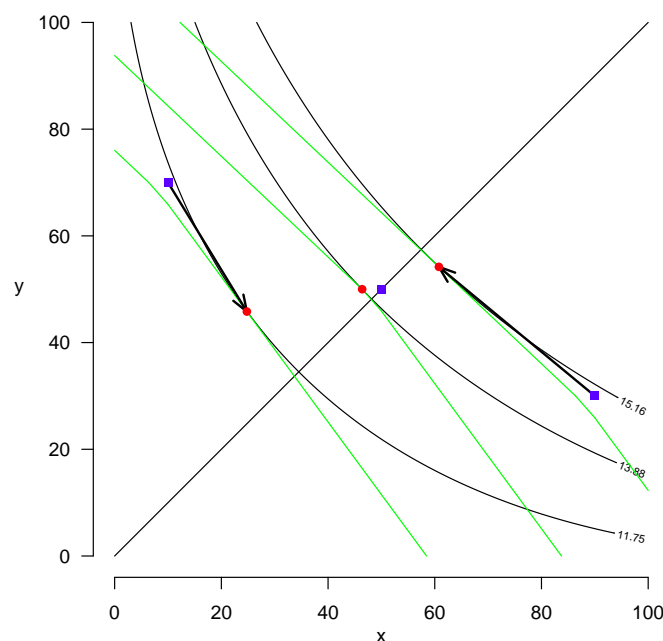
is elected which wishes to tax and spend. Imagine, for example, that there are food shortages on a neighbouring island, and the government decides to raise a poll tax of 12 groats and spend all the tax revenue on apples to be given to the neighbour. We can think of this as an extension to a market with 4 traders,  $A$ ,  $B$ , and  $C$  starting with 30 tokens and a debt of 12 groats, and the government starting with credit of 36 groats. The analysis proceeds as before except that buying both apples and pears is no longer a possible strategy for a trader. In fact it must be replaced by the strategy of selling both goods in sufficient quantities to clear the debt to the government. With this change, we find that we must have  $p = 1.661$  and  $q = 1.466$ .

We then find the revised demands for goods shown in Table 6.

**Table 6:** Equilibrium demands (groats) with taxation

Good	Trader				Total
	$A$	$B$	$C$	$G$	
Apples	-48.55	-6.00	24.55	30.00	0
Pears	35.45	0.00	-35.45	0.00	0
Tokens	1.09	-6.00	-1.09	6.00	0
Total	-12	-12	-12	36	0

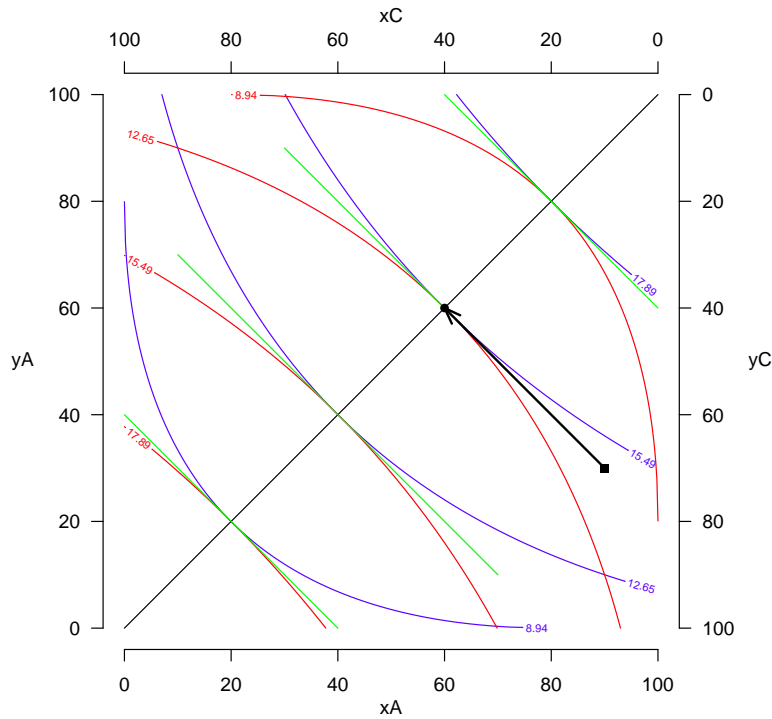
In this solution, trader  $B$  still maximises his utility at a corner point of his budget set (he sells apples and tokens). We can recalculate Figure 1.



**Figure 6:** Movements to market equilibrium with taxation

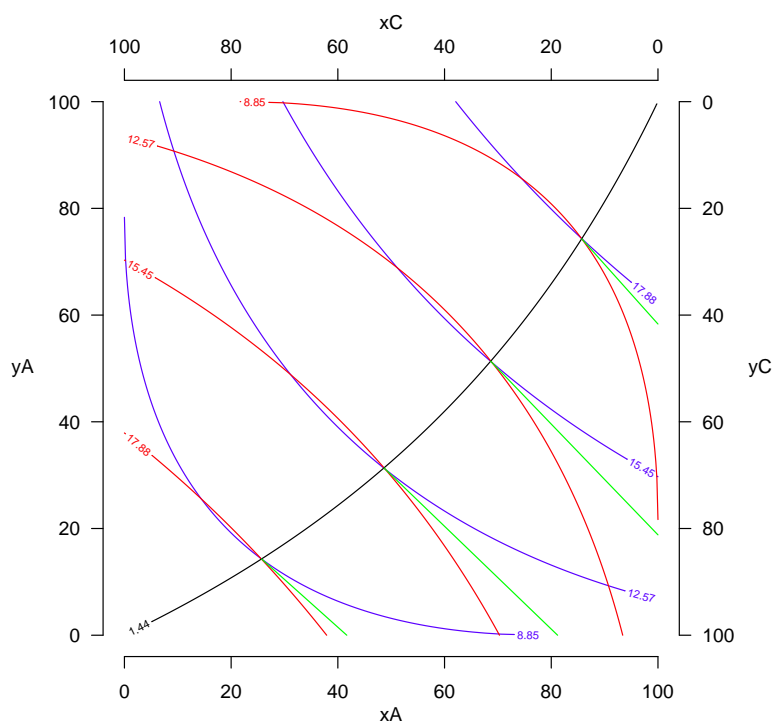
## 4 The Edgeworth box

Let us modify our example by removing trader  $B$ . With just two traders and two commodities, if we do not use tokens we can represent the market prices in an Edgeworth box. Figure 7 shows the utility contours for the two traders (the second trader uses rotated axes). The contract curve (the locus of Pareto efficient points, here the line  $y = x$ ) is the curve where the two contours have a common tangent. These tangents are shown as green lines. (Some authors restrict the contract curve to efficient points which are better than the initial position for both traders.) A point on the contract curve such that the common tangent line passes through the original holdings for the two traders is an equilibrium solution with the relative prices shown by the gradient of the tangent.



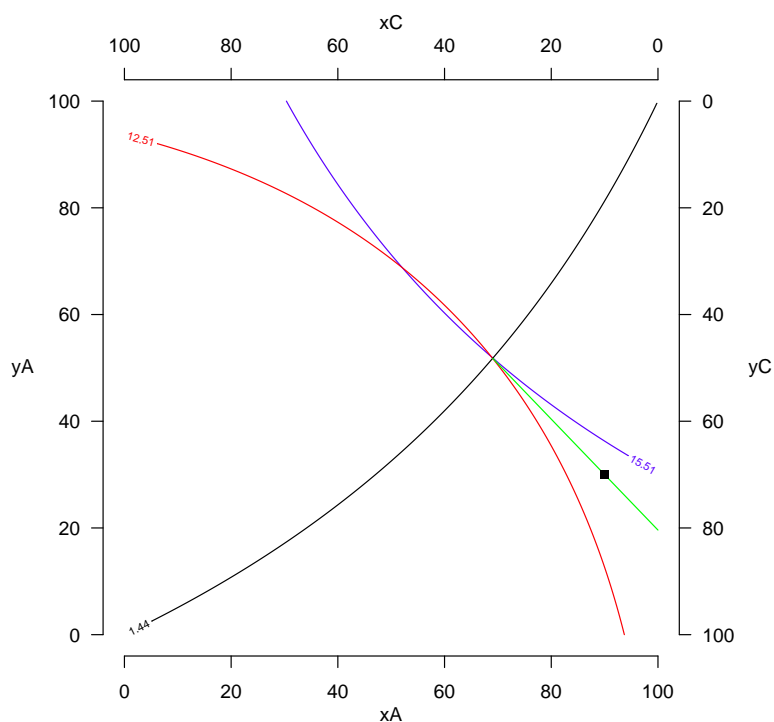
**Figure 7:** Edgeworth box

How does this diagram change when we use tokens with the given values for  $r$  (20%) and  $n$  (30)? We saw that each trader will have three budget constraints: Figure 3 shows these for trader  $A$ , but clearly with just two traders we can find a solution only if  $A$  moves to somewhere along the upper blue line and correspondingly  $C$  to somewhere on the lower red constraint. If the prices of the goods are  $p$  and  $q$ , the blue line has slope  $p/(q(1+r))$  and the red line  $p(1+r)/q$ . The ratio of the slopes is  $(1+r)^2$ . So for a point to be a possible solution, the slopes of the two utility curves at that point must have this ratio. So we replace the contract curve with a revised curve connecting points having this property, as shown in Figure 8.



**Figure 8:** Revised Edgeworth box

The green lines in the figure have slopes which are the geometric mean of the gradients of the two utility contours on the revised contract curve. We seek a point on the revised contract curve such that the corresponding green line passes through the players' starting position. Figure 9 shows what we find.



**Figure 9:** Solving revised Edgeworth box

**Theorem 4.1.** Suppose  $F = (f_1, f_2)$  is a point in the Edgeworth box at which the gradient of the utility curve of the first player (A) is  $-g$ , and of the second player (C) is  $-h$ , and such that  $h/g = (1 + r)^2$  for some  $r > 0$ . Suppose that  $S = (s_1, s_2)$  is a starting position for the two players with  $s_1 > f_1$  and  $s_2 < f_2$  and satisfying

$$\frac{f_2 - s_2}{f_1 - s_1} = -\sqrt{gh}.$$

Then

- (i) the market with tokens having price  $r$  has an equilibrium at  $F$  in which the prices  $p$  and  $q$  for goods 1 and 2 (respectively) satisfy  $p/q = \sqrt{gh}$ ;
- (ii) if each trader is initially given  $n$  tokens, the equilibrium prices will be

$$p = \frac{n}{s_1 - f_1}, \text{ and}$$

$$q = \frac{n}{f_2 - s_2}.$$

*Proof.* The suggested prices are both positive, so we can calculate the optimum sales/purchases for each trader at these prices. Supposing the first trader plans to sell  $-x$  of good 1 and buy  $y$  of good 2. He will gain  $-px$  from the sale and he has  $n$  tokens worth  $rn$ , so he can afford

$$y = \frac{rn - px}{q(1 + r)}.$$

So if he sells  $s_1 - f_1$  of good 1 at the given prices he will be able to buy  $f_2 - s_2$  of good 2. This would move him from point  $S$  to  $F$ . We know that

$$\frac{p}{q} = \frac{f_2 - s_2}{s_1 - f_1} = \sqrt{gh}.$$

We also know that  $h = (1 + r)^2 g$ , so that

$$\frac{p}{q} = (1 + r)g.$$

But we saw that when the first trader sells good 1 in order to buy good 2 he trades along the line

$$px + q(1 + r)y = rn,$$

which has slope

$$-\frac{p}{q(1 + r)} = -g,$$

the slope of his utility contour at  $F$ . Hence the first trader achieves his maximum possible utility (at the assumed prices) at  $F$ . And similarly for the second trader. So we have found an equilibrium.  $\square$

The corollary would be entirely obvious in a system without tokens.

**Corollary 4.1.1.** The equilibrium solution involves each trader paying exactly as much to buy one good as he receives from selling the other.

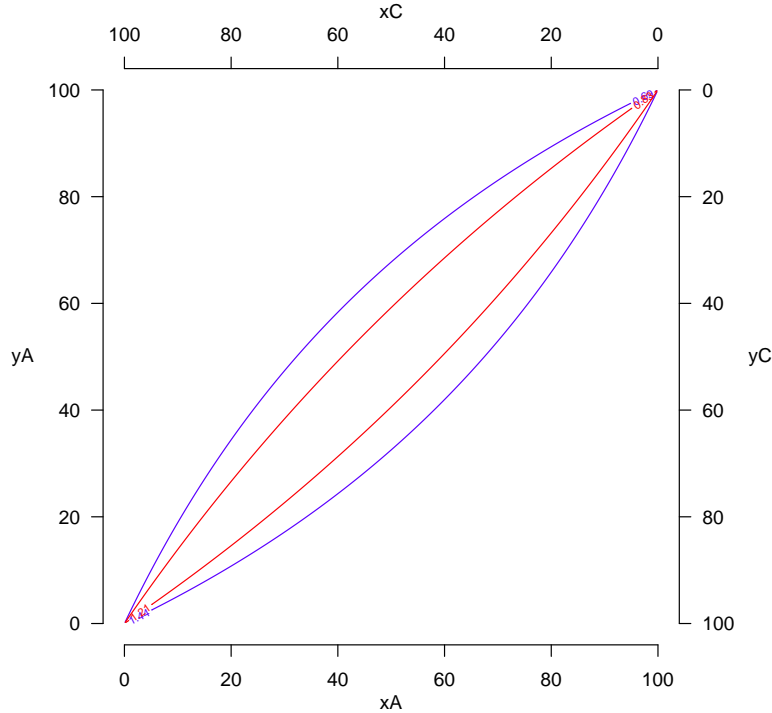


*Proof.* At the equilibrium player  $A$  receives  $p(s_1 - f_1)$  from selling good 1, and pays  $C$  exactly the same amount ( $q(f_2 - s_2)$ ) to purchase good 2.  $\square$

So each is left with  $n$  tokens after these trades. They each owe the bank  $rp(s_1 - f_1) = rq(f_2 - s_2) = rn$ , the value of  $n$  tokens.

In our example,  $F = (69.03, 51.80)$ . At  $F$ ,  $g = 0.866$ , and  $h = 1.247$  so that  $h/g = 1.44 = 1.2^2$ . The geometric mean of the two slopes is 1.040. This is equal to the slope between  $S = (90, 30)$  and  $F$ , so Theorem 4.1 applies, and there will be an equilibrium with  $p = 30/20.97 = 1.43$  and  $q = 30/21.80 = 1.38$ . Player  $A$  will sell 20.97 apples and buy 21.80 pears. She receives 30 groats for the apples, and she started with tokens worth 6 groats. The pears cost her 30 groats (paid to the seller), but she must give 30 tokens (worth 6 groats) to the bank. Her budget is balanced. Similarly for trader  $C$ .

Clearly there will be a second revised contract curve lying above the line  $y = x$  where  $h/g = 1/(1+r)^2$ . Figure 10 shows the two curves as blue lines. The red lines show the corresponding curves when  $r = 1/10$ .



**Figure 10:** Solution regions for  $r = 1/5$  and  $r = 1/10$

In the lens-shaped region between the two curves there will be no equilibrium solution (or at least none with positive prices). We noted this possibility in Section 3.1.

## 5 Two periods

Suppose that the group of people is now told that in fact it will be two weeks before they are rescued. After one week there will be an airdrop of fresh supplies, again with

one box of supplies for each person. The boxes will be individually assigned, so once again the traders will have property rights. They need to trade in the second period as in the first, although the goods they are trading may have changed.

The token system can easily cope with this. A fresh supply of tokens is issued by the bank at the start of the second week. However, there is now a possible difficulty. Traders who are rich in the first week may decide to insure against being poor in the second week by not using all their tokens in the first week. This would result in less tokens being available in week one and more in week two. So the price of goods would be lower in the first week than in the second. Realising this, sensible rich traders in week one should not hoard tokens, but rather lend them to poor people, the debt to be cleared in the second week. Then in each period the same number of tokens is used, although there is some further redistribution of wealth at the start of the second week.

This process would be helped if the finance committee and the central bank announce that they plan to control prices so as to produce a slight inflation in the second period. The idea would be to keep  $r$  the same but make a small increase in  $n$  for week two. Everyone would then anticipate a small increase in prices, and money markets would develop allowing rich people to save money in the first period and have it repaid with interest in the second. So all the tokens issued in the first period are used in the first period (none will be hoarded ‘under the bed’), and the central bank retains firm control of the price level in each period.

An alternative approach would be to give tokens a limited life, so that week one tokens have no value in week two. Either way, it seems a good idea to ensure that money is never a store of value, and the use of controlled inflation seems simpler.

Clearly this system can be extended to more than two time periods, or even to time without end.

## 6 Related work

Before the rigour of general equilibrium theory, the idea that the quantity of money determines prices dates back at least to Copernicus, and was notably revived and restated in the last century by Milton Friedman. An alternative idea that fiat money has value because the government will accept it for the payment of taxes goes back at least to Adam Smith, and is supported by Starr (2013), who relates it to general equilibrium theory and to the more recent trading post games, and gives a clear account of current work. Even in his seminal paper Hahn (1965) suggested a solution to his own problem. Suppose all traders wish to insure against not being able to make the trades they want at the auctioneer’s suggested market prices. Suppose they can only insure with the government, and the government redistributes the proceeds of its insurance business to the traders. Then under certain assumptions an equilibrium will exist with money having a non-zero value. But Hahn did not see why, even supposing traders wished to take out such insurance, agents other than the government could not offer the same service. He was thinking in terms of a model rather than a mechanism.

von Weizsäcker (1974), Howard (1976), and Lerner (1979) (see also Lerner and Colander (1980)) all suggested introducing tradeable permits or certificates to control either price levels (von Weizsäcker) or value-added (Howard and Lerner). Layard (1982) advocated a counter-inflation tax. These proposals all took a macro-economic viewpoint.

At about the same time, at the micro-level, Shapley and Shubik (1977) introduced the idea of replacing a Walrasian market with a non-cooperative game (a Strategic Market Game) where players exchange money for goods at trading posts. There is now an extensive literature on these games. Especially relevant to our mechanism are the papers by Dubey and Geanakoplos (1992, 2003, 2006). These describe an SMG in which price levels are determinate. Traders start with commodities and ‘outside money’, but if they need more money they can borrow ‘inside money’ from a central bank. After using the money (the sole medium of exchange) to trade, they must repay what they have borrowed to the central bank plus an interest charge. In equilibrium, all the outside money must leave the system as interest payments, so if an interest rate is set, prices will have to adjust to make this happen. There are obvious differences between the Dubey and Geanakoplos papers and our system. They are trying to model an economy; we are suggesting a mechanism. They use trading posts; we do not. We provide a new economic control variable (the number of tokens issued per head); they do not. They insist traders use real money, of which there is a limited supply and which can be used only once in a time period; we allow traders to create their own money, and allow money to circulate.

However there are also significant similarities between their system and ours (and also Hahn’s). Our tokens play the same role as the outside money, and our purchase tax percentage mirrors the interest rate set in the Dubey and Geanakoplos model or the insurance premium in Hahn’s story. But note that with our system, a futures market in tokens could determine an interest rate separate from the inflation rate controlled by the central bank.

## **7 Conclusions**

We have shown that it is possible to devise a system that gives a determinate price level without relying on any one good as numéraire, even in a one-period case without access to external money. The scheme gives the financial authorities extra levers with which to control the economy. Whether the idea can be developed as a practical tool (either in small communities or in large economies) is for others to judge.

## References

- DUBEY, P. AND J. GEANAKOPOLOS (1992): "The value of money in a finite-horizon economy : a role for banks," in *Economic analysis of markets and games : essays in honor of Frank Hahn*, ed. by P. Dasgupta, D. Gale, O. Hart, and E. Maskin, Cambridge: MIT Press.
- (2003): "Monetary equilibrium with missing markets," *J. Math. Econom.*, 39, 585–618, special issue on strategic market games.
- (2006): "Determinacy with nominal assets and outside money," *Econom. Theory*, 27, 79–106.
- HAHN, F. H. (1965): "On Some Problems of Proving the Existence of an Equilibrium in a Monetary Economy," in *The Theory of Interest Rates*, ed. by F. H. Hahn and F. P. R. Brechling, St Martin's Press, 126 – 35.
- HOWARD, J. V. (1976): "A Method of Controlling Inflation," *Economic Journal*, 86, 832–44.
- LAYARD, R. (1982): *More Jobs, Less Inflation*, London : Grant McIntyre.
- LERNER, A. P. (1979): "MAP: the market mechanism cure for stagflation," *Atl. Econ. J.*, 7, 12–20.
- LERNER, A. P. AND D. C. COLANDER (1980): *MAP : a market anti-inflation plan*, New York : Harcourt Brace Jovanovich.
- SHAPLEY, L. S. AND M. SHUBIK (1977): "Trade using one commodity as a means of payment," *Journal of Political Economy*, 85, 937–968.
- STARR, R. M. (2013): *Why Is There Money?: Walrasian General Equilibrium Foundations of Monetary Theory*, Edward Elgar Pub.
- VON WEIZSÄCKER, C. C. (1974): *Political limits of traditional stabilization policy*, vol. 19 of *Working papers: Institut für Mathematische Wirtschaftsforschung*, Rheda-Wiedenbrück.